Sources for Gravitational Fields and Cosmology

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Abstract

Analysis of the gravitational source for the Schwarzschild metric indicates that the time and the radial components of the energy momentum tensor are equal. Imposing such a condition on cosmology, we propose a cosmological model that is a modification of the Friedman-Robertson-Walker (FRW) universe. An accelerating universe emerges as a natural consequence of this ansatz.

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I. INTRODUCTION

The gravitational source for the Schwarzschild metric is computed in this article. It is a delta-function at the origin, as expected. Unexpected, however, is a relationship between the time component and the radial component of T^μ_ν. This relationship is explored for a mass shell distribution as well as for a continuous mass distribution.. Based on this characteristic of a gravitational source, the author has applied such a constraint to cosmology. The spacetime that emerges is a modification of the FRW universe. This spacetime inevitably leads to a universe with accelerating expansion. Consistency of the model with other cosmological data, such as that for CMR (cosmic background radiation) and for the accelerating expansion from supernova analysis is examined.

II. THE GRAVITATIONAL SOURCE FOR THE SCHWARZSCHILD METRIC

The Einstein equation for the spherically symmetric and static metric,

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1}$$

is expressed as

$$G^0_0 = (\frac{\lambda(r)'}{r} - \frac{1}{r^2})e^{-\lambda(r)} + \frac{1}{r^2} = 8\pi G T^0_0 = 8\pi G\rho, \tag{2}$$

$$G^1_1 = (-\frac{\nu(r)'}{r} - \frac{1}{r^2})e^{-\lambda(r)} + \frac{1}{r^2} = 8\pi G T^1_1 \tag{3}$$

and

$$G^2_2 = -\frac{1}{2}(\nu(r)'' + \frac{(\nu(r)')^2 - \nu(r)'\lambda(r)'}{2} + \frac{\nu(r)' - \lambda(r)'}{r})e^{-\lambda(r)} = 8\pi G T^2_2 \tag{4}$$

with

$$G^3_3 = G^2_2, \text{ and } T^3_3 = T^2_2. \tag{5}$$

In the source free region, one gets the Schwarzschild solution,

$$e^{\nu(r)} = e^{-\lambda(r)} = 1 - r_s/r, \tag{6}$$

where $r_s = 2GM$ is the Schwartzschild radius.

In order to find the gravitational source, T^μ_ν, for the Schwarzschild metric, one takes the limit

$$e^{\nu(r)} = e^{-\lambda(r)} = \lim_{\alpha\to -3} (1 - (r_s/r)^{-\alpha-2}). \tag{7}$$

Substituting Eq. (7) in Eqs.(2-4), one gets

$$G^0_0 = G^1_1 = (r_s)^{-\alpha-2}(\alpha+3)r^\alpha, \tag{8}$$

and

$$G^2_2 = G^3_3 = \frac{\alpha+2}{2}(r_s)^{-\alpha-2}(\alpha+3)r^\alpha = (\frac{r}{2}\frac{d}{dr}+1)G^0_0. \tag{9}$$

Using the identity,

$$\triangle\frac{1}{r} = \lim_{\alpha\to -3}(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr})r^{\alpha+2} = \lim_{\alpha\to -3}(\alpha+2)(\alpha+3)r^\alpha = -4\pi\delta(\mathbf{x}), \tag{10}$$

or equivalently,

$$\lim_{\alpha\to -3}(\alpha+3)r^\alpha = 4\pi\delta(\mathbf{x}), \tag{11}$$

one obtains the gravitational source functions,

$$T^0_0 = T^1_1 = M\delta(\mathbf{x}), \tag{12}$$

and

$$T^2_2 = T^3_3 = (\frac{r}{2}\frac{d}{dr}+1)M\delta(\mathbf{x}) = (\frac{r}{2}\frac{d}{dr}+1)T^1_1. \tag{13}$$

Since a limit, Eq. (11), is involved, caution is needed when other limits are introduced. For example, one may deduce from the middle expression of Eq. (9) that

$$T^2_2 = T^3_3 \simeq -\frac{1}{2}M\delta(\mathbf{x}) \tag{14}$$

In fact, the conservation law,

$$\nabla_{;\mu}T^\mu_\nu = 0, \tag{15}$$

can be expressed as

$$\nabla_{;\mu}T^\mu_1 = \frac{d}{dr}T^1_1 + \Gamma^\mu_{\mu 1}T^1_1 - \Gamma^\tau_{\mu 1}T^\mu_\tau \tag{16}$$

$$= (\frac{d}{dr}+\frac{2}{r})T^1_1 - \frac{1}{r}(T^2_2+T^3_3) - \frac{1}{2}(\nu(r)'T^0_0 + \lambda(r)'T^1_1), \tag{17}$$

where the Christoffel symbols of the second kind for the Schwarzschild metric,

$$\Gamma^0_{01} = \frac{1}{2}\nu(r)',\ \Gamma^1_{00} = \frac{1}{2}\nu(r)'e^{\nu(r)-\lambda(r)},\ \Gamma^1_{11} = \frac{1}{2}\lambda(r)',\ \Gamma^1_{22} = -re^{-\lambda(r)}, \tag{18}$$

$$\Gamma^1_{33} = -re^{-\lambda(r)}\sin^2\theta,\ \Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r},\ \Gamma^2_{33} = -\sin\theta\cos\theta,\ \Gamma^3_{23} = \cot\theta, \tag{19}$$

have been used. With the conditions,

$$T^0_0 = T^1_1,\ T^2_2 = T^3_3 \tag{20}$$

and

$$\nu(r)' + \lambda(r)' = 0, \tag{21}$$

Eq. (17) and Eq. (12) yield Eq. (13), but not Eq. (14). The rest of Eq. (15) for $\nu = 0, 2$ and 3 is trivially satisfied.

In order to avoid the limiting procedure in Eqs. (7-11), we consider the Schwarzschild metric for a spherical shell mass distribution at $r = r_1$,

$$e^{\nu(r)} = e^{-\lambda(r)} = 1 - (r_s/r)\theta(r - r_1). \tag{22}$$

Substituting Eq. (22) in Eq. (2-4), one obtains

$$T^0_0 = T^1_1 = \frac{M}{4\pi r_1^2}\delta(r - r_1) \tag{23}$$

and

$$T^2_2 = T^3_3 = \frac{M}{8\pi r}\delta'(r - r_1) = \frac{M}{8\pi r_1^2}(r_1\delta'(r - r_1) + \delta(r - r_1)) \tag{24}$$

$$= (\frac{r}{2}\frac{d}{dr} + 1)T^1_1. \tag{25}$$

Here the relationships among T^μ_ν obtained in Eqs. (23-25) are identical with those in Eqs. (12, 13).

It is now easy to extend the above result to a multi-shell mass distribution and finally a continuous mass distribution, since with the Schwarzschild type solution,

$$e^{\nu(r)} = e^{-\lambda(r)}, \tag{26}$$

the Einstein tensors reduce to linear expressions in $e^{-\lambda(r)}$,

$$G^0_0 = G^1_1 = -\frac{(e^{-\lambda(r)})'}{r} - \frac{1}{r^2}(e^{-\lambda(r)} - 1) \tag{27}$$

and

$$G^3_3 = G^2_2 = -\frac{1}{2}(e^{-\lambda(r)})'' - (e^{-\lambda(r)})'. \tag{28}$$

For a multi-shell mass distribution with mass M_i at $r = r_i$, the solution of the Einstein equation can be expressed as

$$e^{\nu(r)} = e^{-\lambda(r)} = 1 - \frac{1}{r}\sum_i 2GM_i\theta(r - r_i), \tag{29}$$

and the gravitational source becomes

$$T_0^0 = T_1^1 = \sum_i \frac{M_i}{4\pi r_i^2}\delta(r - r_i) \tag{30}$$

and

$$T_2^2 = T_3^3 = \sum_i \frac{M}{8\pi r}\delta'(r - r_i) = (\frac{r}{2}\frac{d}{dr} + 1)T_1^1. \tag{31}$$

The extension to a continuum mass distribution is straightforward and the relationships among the gravitational source functions, T_ν^μ, are kept the same: In fact, the metric is given by

$$e^{\nu(r)} = e^{-\lambda(r)} = 1 - \frac{2G}{r}\int_0^r 4\pi\rho(r)r^2dr \tag{32}$$

and the gravitational source functions are expressed as

$$T_0^0 = T_1^1 = \rho \tag{33}$$

and

$$T_2^2 = T_3^3 = (\frac{r}{2}\frac{d}{dr} + 1)T_1^1. \tag{34}$$

For a constant density distribution, the above equations become

$$T_0^0 = T_1^1 = T_2^2 = T_3^3 = \rho = const. \tag{35}$$

This is equivalent to a cosmological constant for the region of constant matter distribution, but it differs from the latter in that it vanishes outside the matter distribution and the constant value is the matter density itself.

III. IMPLICATION OF THE GRAVITATIONAL SOURCE FUNCTIONS

Although we have studied the gravitational source for a spherically symmetric and static metric, the result obtained ,

$$T_0^0 = T_1^1 = \rho \tag{36}$$

and

$$T_2^2 = T_3^3 \neq T_1^1, \tag{37}$$

is unexpected and different from the gravitational sources that we are used to dealing with in many gravitational problems. In the language of an ideal fluid, it corresponds to a negative pressure in the radial direction and it is anisotropic. It resembles to a string to some extent[1] but the string is stretched in the radial direction..It differs from a string in that it has nonvanishing angular components. Maybe this is the nature of a source of gravity. The cosmological data especially indicates the neccesity for negative pressure. The language used in the data analysis is that there is dark energy that defies normal comprehansion. Maybe this is a hint of an inappropriate application of the formalism in cosmological arguments. In particular the nature of the gravitational source, Eqs. (36, 37), has not been utilized. We call a gravitational source that satisfies the constraints, Eqs. (36) and (37), quasi-static. They are satisfied by static metrics, but are also satisfied by a large set of nonstatic metrics.

In the FRW metric, the gravitational source in the matter dominated era is assumed to be

$$T_0^0 = \rho,\ T_1^1 = T_2^2 = T_3^3 = -p = 0. \tag{38}$$

This condition is very different from the condition expressed in Eqs. (36, 37). Although the latter is obtained for a spherical and static metric, the gravitational source in cosmology in the matter dominated era cannot be so much different from it in a comoving frame. By neglecting constraint for the gravitational source such as Eqs. (36, 37), one may miss an important element in the discussion of gravitational phenomena. In the following sections, we develop a cosmological theory based on these constraints.

IV. A FRW MODEL WITH QUASI-STATIC CONSTRAINT

In the framework of the FRW universe, the equality,

$$T_1^1 = T_2^2 = T_3^3, \tag{39}$$

is assumed from the outset. Therefore, the constraints of Eqs. (36) and (37) are outside of the FRW universe. This implies that an extra parameter is introduced as a measure of the departure from the FRW universe. This is an attractive feature for fitting the observed data. However, in this section, we impose Eq. (39), so that the well known FRW framework can

be utilized for the analysis. If agreement with the observed data is not satisfactory, we have to go back the original constraints, Eqs. (36) and (37), introducing an extra parameter.

Let us assume that the gravitational source in the matter dominated era is a mixture of an ideal fluid and quasi-static matter. Then, the gravitational source becomes

$$(T^0_0, T^1_1, T^2_2, T^3_3) = (1-f)(\rho, 0, 0, 0) + f(\rho, \rho, \rho, \rho) = (\rho, f\rho, f\rho, f\rho) \ , \tag{40}$$

where f stands for a constant representing the fraction of the quasi-static component. The case, $f = 1$ and ρ =constant, corresponds to a cosmological constant with mass density ρ, but differs from it in that T^μ_ν vanishes outside the mass distribution. The case, $f = 0$, corresponds to a pure ideal fluid.

The Einstein equation in the FRW framework with the gravitational source, Eq. (40), yields

$$H^2 = (\frac{da/dt}{a})^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2} \tag{41}$$

$$\frac{d^2a/dt^2}{a} = 4\pi G\rho(f - 1/3), \tag{42}$$

with the conservation law,

$$\frac{d(\rho a^{3(1-f)})}{da} = 0, \tag{43}$$

where $k = -1, 0, 1$ corresponds to an open, flat and closed universe, respectively. From Eq. (42), it is clear that one gets an accelerating universe for $f > 1/3$ .Using the solution of Eq. (43),

$$\rho = \frac{A}{a^{3(1-f)}}, \tag{44}$$

where A is an integration constant, one gets the equation for the scale factor $a(t)$,

$$(da/dt)^2 = (8\pi GA/3)a^{3f-1} - k. \tag{45}$$

The solution of Eq. (45) is

$$a(t) \approx e^{\kappa t} \qquad for \ \ f = 1, \tag{46}$$

where

$$\kappa = (8\pi GA/3)^{1/2}, \tag{47}$$

and

$$a(t) \approx (\kappa t)^{(2/3)/(1-f)} \qquad for \ \ 1/3 < f < 1. \tag{48}$$

The solutions indicate that the acceleration is exponential for $f = 1$ and a power law for $1/3 < f < 1$.

In order to compare our result with that of a standard model with a cosmological constant, Λ, we define

$$\Omega_M = 8\pi G\rho/(3H^2) \tag{49}$$

$$\Omega_k = -k/(a^2H^2) \tag{50}$$

and

$$\Omega_\Lambda = \Lambda/(3H^2). \tag{51}$$

Then, Eqs. (41) and (42) can be written

$$1 = \Omega_M + \Omega_k \tag{52}$$

and

$$(da/dt)^2/(aH^2) = (3f - 1)\Omega_M/2. \tag{53}$$

On the other hand, the standard FRW universe with a cosmological constant and $k = 0$ (flat) yields

$$H^2 = (\frac{da/dt}{a})^2 = \frac{8\pi G\rho}{3} + \Lambda/3, \tag{54}$$

$$\frac{d^2a/dt^2}{a} = -4\pi G\rho/3 + \Lambda/3, \tag{55}$$

and the conservation law

$$\rho = \frac{A}{a^3}. \tag{56}$$

From Eqs. (54) and (56), one gets an accelerating solution

$$a(t) \approx e^{(\Lambda/3)^{1/2}t}, \tag{57}$$

and the Ω-relationships

$$1 = \Omega_M + \Omega_\Lambda \tag{58}$$

and

$$(da/dt)^2/(aH^2) = -\Omega_M/2 + \Omega_\Lambda. \tag{59}$$

It is interesting to observe that the parameter set[2][3],

$$\Omega_M = 0.4, \;\; \Omega_\Lambda = 0.6 \tag{60}$$

in Eqs. (58) and (59) and that in Eqs. (54) and (55),

$$\Omega_M = 0.4,\ \ \Omega_k = 0.6,\ \ f \approx 1 \tag{61}$$

give identical results for the Ω-relationships. If one chooses the parameter set,

$$\Omega_M = 0.3,\ \ \Omega_\Lambda = 0.7, \tag{62}$$

then the corresponding parameter set in Eqs. (54) and (55) is

$$\Omega_M = 0.3,\ \ \Omega_k = 0.7,\ \ f = 1.56. \tag{63}$$

The value of f in Eq. (63) indicates that a departure from the FRW framework may be needed if the parameter set Eq. (62) is chosen by the data analysis.

It should also be pointed out that the option Eq. (61) or Eq. (63) requires an open universe. This is definitely in disagreement with the power spectrum data for CMR anisotropy. The data strongly supports a flat universe in the FRW spacetime. Therefore, the model discussed in this section is not a viable option for describing the observed data. In the following section, we introduce a spacetime that is outside of the FRW framework. After all, the quasi-static constraint, Eqs. (36) and (37), requires such an extension of the spacetime.

V. A NON-FRW MODEL WITH QUASI-STATIC CONSTRAINT

In order to establish a spacetime that satisfies the quasi-static constraint, Eqs. (36) and (37), we study a modification of the FRW metric starting with the metric,

$$ds^2 = dt^2 - a(t)^2(e^{F(r)}dr^2 + r^2 d\Omega), \tag{64}$$

and the Einstein equation

$$G^0_0 = 3(\frac{da/dt}{a})^2 + \frac{1}{a^2}(e^{-F(r)}(F(r)'/r - 1/r^2) + 1/r^2) = 8\pi G\rho \tag{65}$$

$$G^1_1 = 2\frac{d^2a/dt^2}{a} + (\frac{da/dt}{a})^2 + \frac{1}{a^2}(1 - e^{-F(r)})/r^2 = 8\pi G\rho \tag{66}$$

$$G^2_2 = G^3_3 = 2\frac{d^2a/dt^2}{a} + (\frac{da/dt}{a})^2 + \frac{1}{a^2}F(r)'e^{-F(r)}/2r = 8\pi G\rho_2, \tag{67}$$

where

$$T^0_0 = T^1_1 = \rho, \tag{68}$$

and

$$T_2^2 = T_3^3 = \rho_2. \tag{69}$$

From Eqs. (65) and (66), it follows that

$$F(r)'e^{-F(r)}/r = 2a\frac{d^2a}{dt^2} - 2(\frac{da}{dt})^2 = A, \tag{70}$$

where A is a constant. The solutions of Eq. (70) are

$$e^{F(r)} = \frac{b}{1-(Ab/2)r^2} \tag{71}$$

and

$$a(t) \approx e^{Bt}, \tag{72}$$

where b and B are integration constants. Normalizing the r variable, one can rewrite the metric

$$ds^2 = dt^2 - a(t)^2(\frac{b}{1-kr^2}dr^2 + r^2 d\Omega), \tag{73}$$

and the Einstein equation

$$G_0^0 = 3(\frac{da/dt}{a})^2 + \frac{1}{a^2}(\frac{3k}{b} + \frac{b-1}{b}\frac{1}{r^2}) = 8\pi G\rho \tag{74}$$

$$G_1^1 = 2\frac{d^2a/dt^2}{a} + (\frac{da/dt}{a})^2 + \frac{1}{a^2}(\frac{k}{b} + \frac{b-1}{b}\frac{1}{r^2}) = 8\pi G\rho \tag{75}$$

$$G_2^2 = G_3^3 = 2\frac{d^2a/dt^2}{a} + (\frac{da/dt}{a})^2 + \frac{1}{a^2}\frac{k}{b} = 8\pi G\rho_2, \tag{76}$$

where $k = -1, 0, 1$, as in the FRW spacetime. Clearly, the case $b = 1$ corresponds to the FRW spacetime in the previous section with $f = 1$ and $\rho = \rho_2$. Combining Eq. (74) and Eq. (75), one gets the solution, Eq. (72), again. From Eqs. (74-76), one gets

$$\frac{\partial}{\partial t}(\rho a^2) = 2\rho_2 a\frac{da}{dt} \tag{77}$$

and

$$\frac{\partial}{\partial r}(\rho a^2) = 2r\rho_2, \tag{78}$$

which are nothing but the conservation law.

An important difference between the FRW spacetime and the spacetime described in this section is that the latter requires a radial dependence for the mass density ρ for $b \neq 1$, but not for ρ_2. Defining

$$\Omega_M = \frac{1}{3}(8\pi G\rho)/H^2, \tag{79}$$

$$\Omega_k = -\frac{k}{ba^2H^2} \tag{80}$$

and

$$\Omega_r = -\frac{b-1}{3ba^2}\frac{1}{r^2H^2} \tag{81}$$

one can rewrite the Einstein equation

$$1 = \Omega_M + \Omega_k + \Omega_r \tag{82}$$

and

$$(da/dt)^2/(aH^2) = \Omega_M + \Omega_r, \tag{83}$$

along with Eqs. (77) and (78). The flatness condition in the FRW spacetime, which is required by the CMR anisotropy data[4], is translated symbolically in our metric as

$$\frac{b}{1-kr^2} = 1 \tag{84}$$

or

$$\frac{b-1}{br^2} = -\frac{k}{b}, \tag{85}$$

which, in turn, yields a constraint,

$$\Omega_r = -\frac{1}{3}\Omega_k. \tag{86}$$

In other words, in our metric, the effects of an open universe, ($k = -1$), and that of $b > 1$ give shifts of the first peak in the anisotropy power spectrum in opposite directions, so that an effective flatness in the FRW spacetime is attained by Eq. (84) or Eq. (86). For example, the parameter set that satisfies Eqs. (82) and (86),

$$\Omega_M = 0.4,\ \Omega_k = 0.9 \text{ and } \Omega_r = -0.3, \tag{87}$$

gives, along with Eq. (83),

$$(da/dt)^2/(aH^2) = 0.4 - 0.3 = 0.1. \tag{88}$$

Clearly, this yields an accelerating expansion.

The radial dependence of the mass distribution in Eqs. (74) and (75) can be utilized to compute the spectral index for fluctuation of the mass distribution

$$\rho(k) = \int \rho(r)e^{i\mathbf{kr}}4\pi r^2 dr \approx \int \frac{1}{r^2}e^{i\mathbf{kr}}4\pi r^2 dr = \frac{4\pi^2}{k} \tag{89}$$

and then

$$P(k) = \frac{\rho(k)}{(2\pi)^2} k^2 \approx k. \tag{90}$$

This is quite consistent with the current measurement[2].

Finally, after an accelerating expansion of the universe, all galaxies may acquire relativistic speed. Then, the quasi-static constraint assumption may not have validity and the value of the parameter f may be reduced below the critical value 1/3 so that the accelerating expansion ceases to exist. In other words, the acceleration of the universe expansion should be a temporary phenomenon after the transition of the radiation dominated era to the matter dominated era. The universe expansion should end up with the phase of a decelerating expansion.

VI. DISCUSSION

If a hybrid model of gravitational sources without the FRW spacetime is adopted, one may use

$$(T^0_0, T^1_1, T^2_2, T^3_3) = (\rho, f\rho, \rho_2, \rho_2). \tag{91}$$

Then, an appropriate metric for the gravitational source, Eq. (91), is given by

$$ds^2 = dt^2 - a(t)^2 \left(\frac{1}{1 - kr^2 - hr^{(1-f)/f}} dr^2 + r^2 d\Omega\right), \tag{92}$$

where $k = -1, 0, 1$, h and f are free parameters. In Eq. (92), $f = 1$ reduces to Eq. (73) in the previous section and $h = 0$ reduces to the FRW spacetime. The new spacetime, Eq. (92), has two parameters, f and h, and it has an advantage to fit the observed data.

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